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Probability and Radical Behaviorism

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The concept of probability appears to be very important in the radical behaviorism of Skinner. Yet, it seems that this probability has not been accurately defined and is still ambiguous. I give a strict, relative frequency interpretation of probability and its applicability to the data from the science of behavior as supplied by cumulative records. Two examples of stochastic processes are given that may model the data from cumulative records that result under conditions of continuous reinforcement and extinction, respectively.

Key words: probability of response, cumulative record, stochastic process, schedule of reinforcement

Probability is a very important concept within Skinner's radical behaviorism (Skinner, 1953). It is the probability of a response that is supposed to be changed as a result of a reinforcement (Skinner, 1957a). Nevertheless, it seems to me that the concept is rather vague and needs to be more explicitly defined; an ostensive definition cannot do. A good summary of various ways in which probability is used in behavior analysis is supplied by Johnson and Morris (1987).

Skinner may have borrowed the probability concept from physics and made it a part of radical behaviorism. He might have seen a useful analogy between the emission of a particle when a radioactive nucleus decays and the emission of a response during operant behavior. Just as, in the former case, there is a probability for the emission of a particle, so, in the latter case, there would be a probability for a response to be emitted.

In *The Behavior of Organisms* (Skinner, 1938), the term "probability" is not found in the index. Instead, he states: "It follows that the main datum to be measured in the study of the dynamic laws of an operant is the length of time elapsing between a response and the response immediately preceding it or, in other words, the rate of responding" (p. 58). But, a few years later Skinner (1944) said: "The business of a science of behavior is to predict response. This prediction is to be achieved by evaluating the strength of a response (the probability that it will occur)" (p. 280). Then, in *Schedules of Reinforcement,* Ferster and Skinner (1957) stated: "Our basic datum is the rate at which such a response is emitted by a freely moving organism" (p. 7), and a few sentences later: "Such a datum is closely associated with the notion of probability of action" (p. 7). Again, Skinner (1957a) stated: "Probability of responding is a difficult datum" (p. 344), and then he went on to appeal to frequency of emission. In a subsection to *Verbal Behavior* (Skinner, 1957b) entitled "Probability and the Single Instance," Skinner said: "Under laboratory conditions probability of response is easily studied in an individual organism as frequency of responding" (p. 28). But, a couple of sentences later he said: "But we need to move on from the study of frequencies to a consideration of the probability of a single event" (p. 28). A few pages back he had said: "Our basic datum is not the occurrence of a given response as such, but the probability that it will occur at a given time" (p. 22). Finally, Skinner (1966) made the following statement: "Many investigators prefer to treat rate of responding as a datum in its own right. Eventually, however, the prediction and control of behavior call for an evaluation of the probability that a response will be emitted" (p. 16).

It seems clear to me that Skinner was trying to find a probability for the next response in time. He believed that probability was to be inferred from the data, but that it was not the same as frequency of responding. Yet, he did not arrive at an exact relationship between the two.

In experiments on operant behavior the data usually consist of a cumulative record, what Skinner (1969) has referred to as the "fact in the bag" (p. 84). The rate of responding could be seen to vary as the slope of that curve varied. This variation was considered an indication of changes in the probability, to be found somewhere in those curves and slopes, for the single instance.

My purpose here is to set forth what I think is the way to apply probability to the data from the science of behavior, specifically that part which is in the form of cumulative records. One result will be a possible solution to Skinner's problem of finding the relation between probability of response and frequency of responding. I think that a complete probabilistic treatment of these data would involve at least the following five steps: first, to supply an unambiguous definition of probability; second, to show how probability applies to the data which have been found by that part of the science of behavior which relies on the cumulative recorder; third, to produce a mathematical model for that data; fourth, to compare the model with experimental results; and fifth, to manipulate given probabilities and then derive other probabilities from them which constitute predictions of the model and should in turn be compared with the results of other experiments. No complete treatment will be given here; only a start is being made with the first three points given above.

I will describe here a definition of probability based on relative frequency, which I consider the only one to be scientifically useful because it is quantitatively unambiguous and directly related to mass phenomena or repeatable physical events. Because relative frequency probability is already predominantly used in behavior analysis (Johnson & Morris, 1987), in the next section I will simply encourage the continuance of that approach. Following that, I will set forth my view of how a connection can be made between probability and that part of the data from the science of behavior that appears in the form of cumulative records. The merits of this connection will apply only to Skinner's use of probability as I have described it above; no claim will be made here as to other uses of probability in the science of behavior generally. Next, I introduce the time-dependent probability distributions, which are called stochastic processes, and their relevance to cumulative records. Finally, I give two explicit examples of stochastic processes that may model the data from the simplest schedules of reinforcement: continuous reinforcement and extinction.

PROBABILITY

The concept of probability in mathematics and physics has a history extending over several centuries and has been, perhaps, the most confused of all physical concepts. Quantitative probability started in the 17th century with the analysis of so-called games of chance, such as rolling dice. To this day, however, there are still at least three different meanings given to probability. Such a state of affairs is rare because concepts will either survive as parts of viable theories or become part of forgotten theories studied only as history of science. I will only summarize those three meanings. A more complete review is given by Johnson and Morris (1987). A fourth meaning of probability, which I discount altogether, is given by Halmos (1944) who says: "Probability is a branch of mathematics" (p. 493). All three concepts of probability that I consider apply to a set of possible outcomes of an experiment. Rolling a die has six possible outcomes; flipping a coin has two. A number is then assigned to each outcome and called its probability. The meaning of that number is a basic difference between the various concepts of probability.

The original meaning as formalized by Laplace (1886) is called classical, a priori, or even logical. It assigns to each simple outcome an equal value based upon the so-called principle of indifference, something like a uniform degree of ignorance or a lack of better information. It would be better to call this approach possibility-counting inasmuch as that is what is involved; the main problem is to find all the "equal" possibilities in the sense of equally probable cases. But, defining "probability" in terms of "equally probable" certainly involves a vicious circle and for that reason alone should be considered unacceptable. Nevertheless, consider an example. With six possible faces on a die and each face having equal probability, it is clear that each will have a value of 1/6. But, it should be clear that such numbers have no physical meaning that could be used to compare with experimental data; it is only the arithmetical result of counting the possible results of an experiment.

It is important to make two further points. First, there is a strong desire on the part of some scientists to give numerical meaning to a probability of single events. Typical of this is the statement of Kemble (1942): "I propose that single-event probability—in particular, that type of single-event probability commonly designated as a priori—should be included despite its nontestability in the list of essential scientific concepts" (p. 16). I do not think that nontestable, and thereby unobservable, concepts would be useful in a science of behavior where the facts are directly observable. Second, attempts have been made to connect these a priori or logical probabilities with observable frequencies via one of the laws of large numbers. For example, Ballentine (1986) says: "From the practical point of view, this is the most important theorem in probability theory, establishing the connection between abstract probabilities and frequencies in observable data" (p. 885). Yet, as far back as the year 1919, von Mises (1981) had already reviewed the import of that theorem, originally derived by Bernoulli, and had shown that such a connection involves a fallacy. It is not possible by means of a mathematical theorem to change the physical interpretation of a symbol used in a theory; if P is a logical probability, it cannot become a relative frequency; it must remain what it was to begin with.

A second approach to probability is called subjective (Keynes, 1973). In this case the probability of each possibility is determined by a wager-type method. This number will then depend on the individual making the wager, which supposedly would be a result of his state of knowledge, which in turn would be a result, perhaps, of introspection. Again it is not possible to check these numbers for any physical consequences. Because this case involves a personal choice of numbers to be picked by an individual, it might be better considered as a problem to be solved by a science of behavior. It may be clear that a gambler is under the control of a VR schedule of reinforcement, but the cause of the particular amount of money put up at one particular bet is certainly multiple, complicated, and different for each gambler. A probabilistic model of cumulative records, it seems to me, must have the probability in the records, not in the observer, so that agreement on measurements of probability can result.

The third approach is called the relative frequency theory of probability. It was first given a rigorous foundation by von Mises (1964) starting around 1919. It was he who introduced the concept of a label, or sample, space for the set of possibilities. The important difference from other approaches is the assumed existence in reality, or at least conceptually, of an indefinitely long sequence of repetitions of an experiment. The resulting denumerably infinite sequence of results, each one an element from the sample space, is called the collective. Without such a collective von Mises refuses to assign a probability. In the case of a die the sample space is: $\{1, 2, 3, 4, 5, 6\}$. A sequence of those integers will constitute a collective for the experiment of tossing a die and might look like: $[6, 3, 3, 1, 5, 6, 2, 4, \cdots]$. The probability of each element of the sample space is defined by: $P_i = \lim_{N \to \infty} N_i/N$, the limiting value of the ratio of the number of times a given element has appeared in the sequence to the total number of experiments. It is assumed that this limit exists; this is the first postulate of the theory. I will not discuss the second postulate that requires the collective to be a random sequence (von Mises, 1964).

The relative frequency definition of probability has been criticized (Williams, 1945) on at least two counts. First, it has been criticized because it requires an infinite sequence that cannot, in reality, be carried out. This criticism ignores the fact that in mechanics the definition of instantaneous velocity along one dimension is $v = \lim_{\Delta t \to 0} \Delta v/\Delta t$ and this also involves a limiting procedure that cannot be carried out in practice. Second, it has been criticized because the theory cannot include within its domain of applicability such uniquely phrased questions as: What is the probability that there is a tenth planet in the solar system? Because there is only one solar system, there is no collective and such a probability is without meaning. Although such questions can be posed in the other approaches, this is not a weakness of the relative frequency theory; it is its delimitation to repeatable or mass phenomena. "Probability" is a form of verbal response emitted under many circumstances; but, within an experimental science, such lay words must become restricted in their usually vague meaning. For example, mechanics has an important concept called "power." Yet mechanics has never been considered incomplete because it is not possible (and would be considered absurd) to assign a numerical value in watts to the power of Congress or the President and thereby resolve the important problem in political science regarding which branch of government is more powerful.

In conclusion, probability is not a property of the single instance but of the whole sequence. When a die is tossed we may say that the probability to get, say, a five is 1/6. That number is only about the whole sequence of tosses, about occurrences of the five in the long run. After some reflection it should become clear that it could not be otherwise. A single toss may produce a six; what then could the 1/6 have indicated, prior to the toss, about the coming event when we did not actually get a five? Nothing.

Once a collective has been defined and a probability distribution assumed, the mathematical theory consists of the techniques whereby new probabilities are deduced from the ones given initially. For example, from the distribution for single tosses of the die one can find the probability distribution of two successive tosses, such as two fives in a row. This is similar to classical mechanics where the initial positions and velocities of particles are assumed given (the initial conditions) and then the theory predicts the future values of those same quantities.

PROBABILITY AND BEHAVIOR

For centuries it has been a philosophical problem to decide whether human actions were determined by causes or were free and the result of free will. I agree with Skinner that human (and animal) behavior is determined or caused in a lawful manner. However, I also think that it is not possible, at least so far, to control behavior sufficiently well that one can predict the single instance. One cannot make, for example, the predictive statement: "At exactly 15 minutes and 36.7 seconds after entering this experimental chamber this pigeon will be pressing this key with its beak." However, it may be possible to say that at the selected time there is a probability that the pigeon will be again pressing the key during the next instant of time. This would mean, in terms of relative frequency probability, that if the experiment were carried out repeatedly over a period of many days and under the same initial circumstances within the same environment, there would be some occasions when the key was being pressed and others when it was not. The fraction of occasions when it was being pressed would constitute the approximate value of the probability of that type of response. I say approximate only because the sequence would necessarily be finite. It is, therefore, not accurate to say "probability of response" because the probability is about the whole class of responses. This class of responses is defined as consisting of those responses that occur in the instant after a given length of time (such as 15 minutes and 36.7 seconds from the beginning of each experiment) has elapsed. A preferable term would be "operant probability." The operant would be the class of responses as defined above.

Consider a large number of cumulative records, each one consisting of the results of one identical session or experiment, and then form the following three groupings: first, the totality of all records; second, those records from the total number that have by time t accumulated n responses or, graphically, those in which a vertical line placed on the cumulative record at time t will have intersected the record at the ordinate value n; third, those from this second group which by time $t + \Delta t$ have a total of $n + 1$ responses or, graphically, those in which a vertical line placed at time $t + \Delta t$ on the cumulative record will have intersected the record at the ordinate value $n + 1$. The ratio of the second group to the first is $P_n(t)$, the probability that n responses have occurred by time t, whereas the ratio of the third to the second group is $P_n(t, t + \Delta t)$, the transition probability that given n responses by time t there will be $n + 1$ responses by time $t + \Delta t$. It is this latter probability that I believe Skinner was trying to identify; it is the probability of the next response after n responses have already been emitted. This probability is a relative frequency amongst repeatable experiments as they have been developing up to some given, common value of the elapsed time as measured from their individual starting points. Rate of response, on the other hand, is a ratio of responses to time interval and has units of inverse time and therefore is not a probability; a relative frequency must be a ratio of two things of the same type. Further on I will show that a connection can be found between transition probability and response rate.

To prepare the way for a mathematical treatment of probability it is useful to first introduce the concept of a random variable. The purpose is to introduce points that are real numbers instead of the generally abstract points of the sample space. For example, flipping a coin has the sample space {heads, tails} or {H, T}. It is more convenient to identify heads with 1 and tails with 0, or any other pair of integers. The same could be said about pressing a key; the sample space consists of the two points: to-press and not-to-press. It is more convenient to identify to-press with the integer 1 and not-to-press with the integer 0. A random variable is defined as a real-valued function with the sample space as its domain. The values of the variable, therefore, have attached to them a probability inasmuch as each point of the sample space has a probability. The values of the random variable occur with a certain probability. This explains why the name "random" is used. If, as is the case with a die, the sample space already consists of real numbers, then one can invoke the identity function. In this way it is always safe to talk about the values of a random variable, knowing they are always real numbers. The set of values a random variable may take is called the state space.

STOCHASTIC PROCESSES

A probability distribution is the assignment of a numerical probability to each point of the sample space. The same can be said for the values taken by a random variable in the state space. I consider only discrete sample and state spaces. I have shown that the probability of a specific behavior, such as pressing a key, depends on time. A family of random variables $X(t)$ depending upon a continuous parameter t, in this case the time, is called a stochastic process. If all the random variables belonging to the family are identically distributed, then the continuous time dependence is irrelevant. If they should instead be totally independent of each other, the mathematical situation would become unwieldy, as well as too general to be interesting or applicable. Therefore, there will always be assumed some kind of dependence amongst the random variables.

It is a stochastic process that I believe models the data produced by those operant behavior experiments that use the cumulative recorder. In such an experiment, the data will form a cumulative record that consists of a graph of the total number of pressing responses plotted along the ordinate versus the time along the abscissa. In terms of relative frequency probability, I consider the total number of responses emitted up to a time t to be a stochastic process $N(t)$. The results of a given experiment over some interval of time will constitute a single cumulative record; this is a realization of the stochastic process and is called a sample function. The set of all the cumulative records resulting from the indefinite repetition of an experiment constitutes the stochastic process; the probability, as I illustrated in the previous section, is a property of the whole set, not of a single record.

The state space over which $N(t)$ takes values is the set of nonnegative integers; the sample functions will be monotonic, increasing step-functions of the time. The probability distributions must satisfy: $P\{N(0) = 0\} = 1$ and $P\{N(t) < N(t')\} = 0$ whenever $t' < t$. The transition probability, $P_n(t, t + \Delta t)$, will depend on Δt and will be considered meaningful only to the first order in Δt. For this reason it is also called the infinitesimal transition probability. It is also convenient to introduce the transition probability density, which is the limit of the ratio $P_n(t, t + \Delta t)/\Delta t$, as $\Delta t \to 0$. Attached to the transition probability is the transition random variable $\Delta N(t) \equiv N(t + \Delta t) - N(t)$, whose state space will consist of only the two integers 1 and 0, corresponding to "press" and "no-press," respectively.

As with any experiment in science, the results of an operant experiment must be reproducible when carried out under the same circumstances. Nevertheless, it is doubtful that such cumulative records if placed on top of each other would coincide point for point. Instead, the individual records are unified and related when one considers them as sample functions of a single stochastic process.

I think that this mathematical model of the data contained in the cumulative records of an operant experiment may again make these records important. Skinner (1976) has decried their scarcity in articles on operant research, whereas Zeiler (1984) has called schedules of reinforcement "the sleeping giant." Killeen (1985) has commented on their unwieldiness and makes the interesting remark that if cumulative records are smoothed out, as Skinner suggests, they become more complex because there are more of them to deal with, infinitely more. What might be needed is a way to give them unity through mathematical form so that they can be meaningfully manipulated, just as it became possible to do with the data from physics experiments during the 17th century. However, it should be very clear that this mathematics is not being used to model hypothetical, internal causes of behavior, only the actual, directly observable data.

With the above model that I have suggested, one can keep the step-ladder shape of the cumulative records without smoothing them out. In fact, any reference to the "slope" of a cumulative record cannot be strictly meaningful as these curves have slopes that only take values of zero or infinity; such records have only average slopes one might produce by "eyeballing" with a ruler. Smoothness may instead be found in the expectation $E[N(t)]$ of the stochastic process $N(t)$. The expectation can be a differentiable function of the time, having a well-defined and continuously varying slope at every point in time. In the following section I give two examples.

EXAMPLES

Continuous Reinforcement

I will give a physically intuitive derivation of the well-known Poisson stochastic process first. This may be the way to characterize a pigeon's key-pressing under a schedule of continuous reinforcement. As I said in the previous section, the family of probability distributions for the stochastic process $N(t)$ must have some degree of interdependence; this will appear through the use of a transition probability. In the present case the basic starting equations are:

$$P_{n+1}(t + \Delta t) = P_n(t)P_n(t, t + \Delta t) + P_{n+1}(t) \cdot [1 - P_n(t, t + \Delta t)], \quad n \geq 0, \qquad (1)$$

$$P_0(t + \Delta t) = P_0(t)P_0(t, t + \Delta t), \qquad (2)$$

$$P_n(t, t + \Delta t) = \mathrm{r}\Delta t, \qquad (3)$$

where I am using the definition

$$P_n(t) \equiv P\{N(t) = n\}, \qquad (4)$$

and r is a constant parameter. Equations 1, 2, and 3 are accurate to the first order in Δt.

An intuitive explanation of the results of these equations follows. Equation 1 states that the probability for $n + 1$ responses by time $t + \Delta t$, the term of the left-hand side of the equation, is the sum of two contributions that constitute the two terms of the right-hand side of the equation. The first term consists of the product of the probability that n responses have occurred by time t times the transition probability that a response occurs during the following interval Δt; whereas the second term is a product of the probability that by time t already $n + 1$ responses have occurred times the transition probability that no response occurs during the following time interval Δt. There are only two contributions and just two terms on the right of the equation because it is assumed that over a short enough interval of time Δt there is a negligible transition probability involving the emission of two or more responses. Equation 2, the only case not covered by Equation 1, states that the probability for no response by time $t + \Delta t$ is the product of the probability that no responses have occurred by time t times the transition probability that no response occurs during the following interval Δt. Equation 3 is the basic assumption specific to this process; it states that the transition probability is independent of both the state n and the time t since the start of the experiment and is directly proportional only to the small interval of time Δt.

Substituting from Equation 3 into Equation 2, rearranging, and going to the limit as $\Delta t \to 0$ results in the differential equation

$$dP_0(t)/dt = -\mathrm{r}P_0(t). \qquad (5)$$

Doing the same thing with Equation 1 results in the differential-difference equation

$$dP_{n+}(t)/dt = \mathrm{r}[P_n(t) - P_{n+1}(t)]. \qquad (6)$$

The solution to Equation 5 is

$$P_0(t) = \exp(-\mathrm{r}t). \qquad (7)$$

And, by induction on n, one gets from Equation 6 and Equation 7 the complete solution

$$P_n(t) = [(\mathrm{r}t)^n/n!] \exp(-\mathrm{r}t). \qquad (8)$$

Using Equation 8 one can calculate the expected value of $N(t)$

$$E[N(t)] = \sum_{n=0}^{\infty} nP_n(t) = \mathrm{r}t. \qquad (9)$$

One can see that the expected value is a smooth function of time, and its first derivative, the slope, is

$$dE[N(t)]/dt = \mathrm{r}. \qquad (10)$$

This shows that even though the sample functions do not have a smooth slope, the expected value does have one. In this case the slope is constant, given by r. The simplicity of the result presented in Equation 9 may be misleading because it seems to show that the expected number of responses is always equal to the product of a response rate r and the time t. It must be kept in mind, however, that r, as introduced above in Equation 3, is a transition probability density and defined at each point in time; it is only because I took it, in this case, to be independent of time that it turns out to be equal to the expected response rate. In the next section the results will be quite different.

It is now possible to find the relationship that exists, at least for this process, between the probability of the next response and the response rate. Equation 10 gives us the expected response rate as

r. Equation 3 shows that the infinitesimal transition probability is $r\Delta t$, or that the transition probability density is r. Therefore, it is the transition probability density that is equal to the expected response rate. This is the connection that Skinner may have been searching for, with the qualifications that: (1) Instead of an average response rate from "eyeballing" a single segment of a cumulative record, it is an instantaneous, expected response rate that is required; and (2) instead of a probability of response, it is a transition probability density that is required. All of this results not from one record but from the records of an indefinite repetition of identically prepared experiments. It is a credit to Skinner's great intuition, no doubt in part due to his close and long-time involvement with the directly observable facts of experiments resulting in large numbers of cumulative records, that he insisted on probability being a primary concept and that it be related to response rate.

The distribution in Equation 8 can be used to make a prediction on the distribution of interresponse times. Let Z_1 be the time interval from the start of the experiment to the first response. Then Z_1 is a random variable and we have

$$P(Z_1 < t) = 1 - P(Z_1 > t) = 1 - P_0(t) = 1 - \exp(-rt). \qquad (11)$$

Therefore, Z_1 has a probability density given by

$$dP(Z_1 < t)/dt = r\exp(-rt). \qquad (12)$$

It can be shown by induction (Bhat, 1984) that the interresponse time Z_n between the $(n - 1)$th and the nth responses is distributed according to the same exponential law of Equation 12,

$$dP(Z_n < t)/dt = r\exp(-rt). \qquad (13)$$

This is a good example of what one does in the theory of probability. From the distributions for $N(t)$ I have been able to find the distributions for Z_n. The expected value of Z_n is given by

$$E[Z_n] = \int dt\, tr\exp(-rt) = 1/r. \qquad (14)$$

Because Equations 13 and 14 show that all Z_n are equally distributed with the same mean value 1/r, it is then possible, in this case, to find from a single sample function an approximate value for the parameter r.

Extinction

In the previous case, r was taken to be a constant and the expected value of $N(t)$ had a graph with a fixed slope. Extinction records show a diminishing average slope, becoming flat as no further responses are emitted due to the lack of reinforcement. I think a good model for such data would be a stochastic process again resulting from Equations 1 and 2, but where the transition probability is given by

$$P_n(t, t + \Delta t) = s(t)\Delta t \qquad (15)$$

instead of by Equation 3. Now the transition probability does depend upon the time t since the start of the experiment but is still independent of the state n. What I think is a good choice for the function $s(t)$ is given by

$$s(t) = Qb/(t + b)^2, \qquad (16)$$

which contains the two parameters Q and b.

As in the previous subsection, it is possible to solve the differential-difference equation that results from introducing Equation 16 in Equations 1 and 2. The solution, $P_n(t)$, is given by

$$P_n(t) = (Q^n/n!)[t/(t + b)]^n \cdot\exp[-Qt/(t + b)]. \qquad (17)$$

In this case the expectation will be

$$E[N(t)] = Qt/(t + b). \qquad (18)$$

And the slope of the expectation is

$$dE[N(t)]/dt = Qb/(t + b)^2. \qquad (19)$$

From Equation 19 one can again see that the expectation has a smooth slope. A comparison of Equations 9 and 18 shows that for this example the expected number of responses is not a simple product of a constant r times the time t. A comparison of Equations 16 and 19, however, shows that again in this case the expected response rate is equal to the transition probability density. The interpretation of the parameters Q and b can also be found. The asymptotic value of the expectation as $t \to \infty$ is found from Equation 18 to be Q. Therefore, Q is the expected value of the total number of responses emitted during the process of extinction. One might be tempted to call this number the reflex reserve, a term once used by Skinner (1938). The initial slope (at $t = 0$) of the expectation is Q/b. This can be interpreted as the constant slope of the expected value of $N(t)$, which existed while on continuous reinforcement just before extinction began. One can also interpret b as the time it would have taken to emit Q responses under continuous reinforcement if extinction had not been imposed. One can look upon b as a "time constant" for nonexponential behavior.

I found an interesting remark by Skinner (1979) in the second volume of his autobiography. Referring to graduate students trying to be mathematical about data he says:

> I offered them the smooth and reproducible extinction curve obtained after a brief exposure to periodic reconditioning. (I performed one mathematical analysis myself in a practical way. I plotted the curve on a large vertical board and held a fine gold chain against it. When I tilted the curve upside down at just the right angle, it was covered by the chain. In other words, the curve was a catenary.) (p. 235)

As has been said above, the curves are actually not smooth; trying to smooth them makes them more intractable. The reproducibility of the curves is only approximate, and in fact I consider them to be the sample functions of a stochastic process whose expected value is a smooth function of time. Although the curve may have looked like a catenary, it could not have been. Even if we accept that single records appear smooth only on the average as a result of "eyeballing," the catenary does not have any asymptotes. Yet, the initial slope, as extinction is started, and the flat slope, as key-pressing ceases, require two asymptotes for the expectation of these sample functions. It was based on this that I decided on the form of Equation 16 for $s(t)$ because this then gives the expectation of Equation 18, which does have two asymptotes. It is quite possible that those early attempts at a mathematical treatment of cumulative records failed because probability did not enter into the interpretation; the records were considered unique rather than the sample functions of a stochastic process.

CONCLUDING REMARKS

As Johnson and Morris (1987) have shown, the concept of probability in behavior analysis varied, with no single point of view. The first point I will make is about the nature of probability. I consider that in operant behavior, as in any experimental science, probability is not about a single response but about a class of responses. Given that the predominant view of probability is that it is a relative frequency (Johnson & Morris, 1987), I am first of all trying to encourage that point of view. My main point, however, is with regard to which relative frequency is to be called probability, and then only in conjunction with cumulative records. I have specifically intended to find a consistent solution to Skinner's use of the term, including the relationship between probability and response rate.

I take what has been called a molecular view rather than a molar one. Behavior is dynamic and requires a development over time; the probability of behavior is therefore time dependent. My suggestion is that the data from experiments in operant behavior, as found in cumulative records, be modeled by a stochastic process. The probability of an operant, such as pressing a key, is the relative frequency with which that type of response occurs at time t when an experiment is indefinitely repeated. Each record is a sample function, and the relative frequency of the particular response to the total number of sample functions is the probability of the operant, or class of responses, occuring at time t. I believe that the stochastic process gives a mathematical unity to the class of records of a given schedule.

The model for a cumulative record is then $N(t)$, the total number of responses up to a given time, taken to be a family of random variables or a stochastic process. Then its expectation $E[N(t)]$ is a smooth curve, and one might expect to find that its slope will be close to what one would figure from "eyeballing" a single record, especially when it comes to trends in the slope, the so-called acceleration. We also get the result that the expected response rate is equal to the transition probability density at each point in time.

I have only included models for two schedules of reinforcement: continuous reinforcement and extinction. These models would now have to be compared with experimental results. Also, it is necessary to look at the more complex case of intermittent reinforcement.

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